

Infinitely many nonlocal conservation laws for the ABC equation with $A + B + C \neq 0$

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We construct an infinite hierarchy of nonlocal conservation laws for the ABC equation $Au_t u_{xy} + Bu_x u_{ty} + Cu_y u_{tx} = 0$, where A, B, C are nonzero constants and $A + B + C \neq 0$, using a non-isospectral Lax pair. As a byproduct, we present new coverings for the equation in question. The method of proof of nontriviality of the conservation laws under study is quite general and can be applied to many other integrable multidimensional systems.

Keywords: integrable systems; conservation laws; Lax pairs.

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1 Introduction

Integrable systems are well known to play an important role in modern mathematics, both pure and applied, see e.g. [1, 4, 6, 7, 8, 9, 10, 11, 14, 15, 16, 17, 32, 34] and references therein. Existence of an infinite hierarchy of conservation laws is among the most important features of integrable¹ systems of partial differential equations [1, 4, 11, 32]. It imposes strong constraints on the associated dynamics making it highly regular.

While such a hierarchy of conservation laws can often be extracted from the Lax-type representation of the system under study, cf. e.g. [4, 11] and references therein, in a relatively straightforward manner, rigorous proof of nontriviality and independence of the conservation laws in question is often a tricky matter, especially in the case of integrable systems of partial differential equations in more than two independent variables when the conservation laws under study often happen to be nonlocal, see e.g. [4, 24].

In the present paper we demonstrate how to prove nontriviality and independence of such nonlocal conservation laws at the example of the ABC equation. The procedure presented below is based on the careful examination of the structure of the kernel of total derivatives and is fairly readily generalized to other multidimensional integrable systems.

Recall that the ABC equation has the form

$$Au_t u_{xy} + Bu_x u_{ty} + Cu_y u_{tx} = 0, \quad (1)$$

where A, B, C are arbitrary nonzero constants (if one of them vanishes, (1) reduces to a first-order PDE).

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¹In the present paper we mean by integrability existence of a nontrivial Lax pair for the system under study, cf. e.g. [1, 11, 39] and references therein for details.

To the best of our knowledge, equation (1) has first appeared in [38] in connection with the study of geometry of Veronese webs (cf. also [25] and references therein). In the same paper the author has established integrability of (1) for the case of $A+B+C=0$ by presenting the associated Lax pair; note that in [31] a four-dimensional integrable generalization of (1) with $A+B+C=0$ was found. Later in [27] (cf. also [28, 30, 36, 37] for related results) a recursion operator for (1) with $A+B+C=0$ was found, and using the method of hydrodynamic reductions it was shown [5] that (1) is also integrable if $A+B+C \neq 0$. Note also that if $A=B=C \neq 0$ then (1) admits [12] a Lagrangian with the density $-Au_x u_y u_t/2$.

Below we assume that $A+B+C \neq 0$ and put $B = -\kappa_1 A$, $C = -\kappa_2 A$, so $\kappa_1 + \kappa_2 - 1 \neq 0$. Then equation (1) takes the form

$$u_{xy} = \frac{\kappa_1 u_x u_{ty} + \kappa_2 u_y u_{tx}}{u_t}. \quad (2)$$

Integrability of (2) is an immediate consequence of the following result which provides a nonlinear Lax pair for the equation in question.

Proposition 1 ([5]). *The ABC equation (2) has a covering defined by the system*

$$\begin{aligned} q_y &= u_y \frac{\kappa_1 s R' + (\kappa_1 + \kappa_2 - 1) R}{\kappa_1 R'}, \\ q_t &= u_t \frac{\kappa_1 s R' + R^2 + (\kappa_1 + \kappa_2 - 1) R}{\kappa_1 R'}, \end{aligned} \quad (3)$$

where $s = q_x/u_x$, and the function $R = R(s)$ is a solution to the ODE

$$R'' = \frac{(2(\kappa_1 + \kappa_2) - 1) R + (\kappa_1 + \kappa_2 - 1)(2\kappa_1 + \kappa_2 - 1)}{(\kappa_1 + \kappa_2 - 1) R (R + \kappa_1 + \kappa_2 - 1)} (R')^2. \quad (4)$$

To make the exposition self-contained, we give a very brief introduction to the theory of differential coverings in Section 2; for more details of this theory we refer the reader to [4, 22, 23, 24] and references therein; the discussion of how certain types of nonlinear coverings are related to integrability can be found e.g. in [11, 29, 39].

Remark 1. Equation (4) is integrable by quadratures. Its general solution is of the form $R(s) = \Omega(c_1 s + c_2)$, where Ω is the inverse function for the function

$$\omega(z) = \int z^{-(2\kappa_1 + \kappa_2 - 1)/(\kappa_1 + \kappa_2 - 1)} (z + \kappa_1 + \kappa_2 - 1)^{-\kappa_2/(\kappa_1 + \kappa_2 - 1)} dz$$

and c_1, c_2 are arbitrary constants.

Most importantly, (3) gives rise² to a linear nonisospectral Lax pair for (2):

Corollary 1. *The ABC equation (2) admits a linear nonisospectral Lax pair of the form*

$$\begin{aligned} \Phi_y &= (H_1)_\zeta \Phi_x - (H_1)_x \Phi_\zeta, \\ \Phi_t &= (H_2)_\zeta \Phi_x - (H_2)_x \Phi_\zeta, \end{aligned} \quad (5)$$

where

$$H_1 = \left(u_y \frac{\kappa_1 s R' + (\kappa_1 + \kappa_2 - 1) R}{\kappa_1 R'} \right)_{s=\zeta/u_x}, \quad H_2 = \left(u_t \frac{\kappa_1 s R' + R^2 + (\kappa_1 + \kappa_2 - 1) R}{\kappa_1 R'} \right)_{s=\zeta/u_x}.$$

The coefficients of (5) depend on the variable spectral parameter ζ in a highly nontrivial fashion thanks to the presence of the function R , so our first order of business is to construct a Lax representation with simpler dependence on the parameter. This is done in Section 3, where we present a new covering for (2) and the associated nonisospectral Lax pair (14) and show how it is related to (5). Finally, in Section 4 we construct an infinite hierarchy of nonlocal conservation laws for (2) and prove their nontriviality. Note that the method of the proof is quite general and can be applied to many other integrable multidimensional systems.

²See e.g. [11, Subsection 10.3.3] and [13] and references therein for the general construction leading from a covering of the type (3) to a Lax pair of the type (5), and [11, 26] and references therein for nonisospectral Lax pairs in general.

2 Differential coverings

We briefly review here the theory of differential coverings over infinitely prolonged differential equations. The reader can find further details and examples in [4, 22, 23, 24].

Let M be a smooth manifold, $\dim M = n$, and $\pi: E \rightarrow M$ be a vector bundle, $\text{rank } \pi = m$. We consider the bundles of k -jets $\pi_k: J^k(\pi) \rightarrow M$, $k \geq 0$, together with the natural projections $\pi_{k+1,k}: J^{k+1}(\pi) \rightarrow J^k(\pi)$. Then the manifold of infinite jets $J^\infty(\pi)$ is defined as the inverse limit with respect to these projections and the bundles $\pi_\infty: J^\infty(\pi) \rightarrow M$ and $\pi_{\infty,k}: J^\infty(\pi) \rightarrow J^k(\pi)$ are defined as well. For any section $s: M \rightarrow E$ of the bundle π its infinite jet $j_\infty(s): M \rightarrow J^\infty(\pi)$ is a section of π_∞ . One has the embeddings $\pi_{k+1,k}^*: C^\infty(J^k(\pi)) \rightarrow C^\infty(J^{k+1}(\pi))$, and we define the algebra of smooth functions on $J^\infty(\pi)$ as $\mathcal{F}(\pi) = \bigcup_{k \geq 0} C^\infty(J^k(\pi))$.

The main geometric structure on $J^\infty(\pi)$ is the Cartan distribution \mathcal{C} : for any point $\theta \in J^\infty(\pi)$ we define the Cartan plane \mathcal{C}_θ as the tangent plane to the graph of an infinite jet passing through this point. This distribution is formally integrable: if X and Y are vector fields lying in \mathcal{C} then the commutator $[X, Y]$ lies there as well. Every Cartan plane \mathcal{C}_θ is n -dimensional and projects isomorphically to $T_{\pi_\infty(\theta)}M$ by the differential of π_∞ . Due to this, any vector field Z on M can be uniquely lifted to a vector field \mathcal{C}_Z on $J^\infty(\pi)$. The correspondence $X \mapsto \mathcal{C}_X$ is $C^\infty(M)$ -linear and preserves the commutator. In addition, $\pi_{\infty,*}(\mathcal{C}_X) = X$. In other words, we have a connection which is called the Cartan connection. In the standard local coordinates $x^1, \dots, x^n, u_\sigma^1, \dots, u_\sigma^m$ in $J^\infty(\pi)$, σ being symmetric multi-index consisting of the integers $1, \dots, n$, the Cartan connection is determined by the correspondence

$$\mathcal{C}: \frac{\partial}{\partial x^i} \mapsto D_{x^i} = \frac{\partial}{\partial x^i} + \sum_{\sigma,j} u_{\sigma i}^j \frac{\partial}{\partial u_\sigma^j}, \quad (6)$$

where the fields D_{x^i} are called the total derivatives. Differential operators in total derivatives are called \mathcal{C} -differential operators.

A differential equation³ of order k is a submanifold in $J^k(\pi)$. Locally, it can be given by the conditions $F^1 = \dots = F^r = 0$, where F^j are smooth functions on $J^k(\pi)$. Its infinite prolongation is the submanifold (probably, with singularities) \mathcal{E} in $J^\infty(\pi)$ satisfying the conditions $(D_{x^{i_1}} \circ \dots \circ D_{x^{i_s}})(F^j) = 0$ for all $j = 1, \dots, r$, $s \geq 0$, and $1 \leq i_1, \dots, i_s \leq n$. The Cartan connection can be restricted from the bundle π_∞ to its subbundle $\pi_\infty|_{\mathcal{E}}: \mathcal{E} \rightarrow M$ and hence any \mathcal{C} -differential operator restricts from $J^\infty(\pi)$ to M . On the other hand, the correspondence (6) allows one to lift any linear differential operator on M to a \mathcal{C} -differential operator on \mathcal{E} . We always assume that \mathcal{E} is differentially connected which means that the only solutions of the system $D_{x^i}(f) = 0$, $i = 1, \dots, n$, on \mathcal{E} are constants.

In particular, let $\ell_{\mathcal{E}}$ denote the restriction of the linearization operator

$$\ell_F = \left(\sum_{\sigma} \frac{\partial F^\alpha}{\partial u_\sigma^\beta} D_\sigma \right), \quad \alpha = 1, \dots, r, \quad \beta = 1, \dots, m,$$

to \mathcal{E} . Then the solutions of the equation $\ell_{\mathcal{E}}(\varphi) = 0$ are identified with symmetries of \mathcal{E} , i.e., with the evolutionary vector fields $\sum_{j,\sigma} D_\sigma(\varphi^j) \partial / \partial u_\sigma^j$ that preserve the Cartan distribution on \mathcal{E} . Dually, the solutions of $\ell_{\mathcal{E}}^*(\psi) = 0$ are called cosymmetries, where $\ell_{\mathcal{E}}^*$ is formally adjoint to $\ell_{\mathcal{E}}$. The lift d_h of the de Rham differential gives rise to the horizontal de Rham complex on \mathcal{E} ; d_h -closed $(n-1)$ -forms are conservation laws of \mathcal{E} and d_h -exact forms are trivial conservation laws. To any conservation law ω one can associate its generating section (or the characteristic) ψ_ω which is a cosymmetry.

A morphism of equations is a smooth map $\tau: \tilde{\mathcal{E}} \rightarrow \mathcal{E}$ which takes the Cartan distribution $\tilde{\mathcal{C}}$ on $\tilde{\mathcal{E}}$ to that on \mathcal{E} . A morphism τ is a (differential) covering if its differential maps the Cartan plane $\mathcal{C}_{\tilde{\theta}}$ to $\mathcal{C}_{\tau(\tilde{\theta})}$ isomorphically for any $\tilde{\theta} \in \tilde{\mathcal{E}}$. In other words, for any vector field Z on M the field $\tilde{\mathcal{C}}_Z$ projects to \mathcal{C}_Z . Locally, this means that the total derivatives on $\tilde{\mathcal{E}}$ are

$$\tilde{D}_{x^i} = D_{x^i} + X_i, \quad i = 1, \dots, n, \quad (7)$$

³By a slight abuse of terminology, we speak of a differential equation even though it could actually be a system of differential equations.

where D_{x^i} are the total derivatives on \mathcal{E} , and

$$D_{x^i}(X_j) - D_{x^j}(X_i) + [X_i, X_j] = 0, \quad 1 \leq i < j \leq n,$$

X_i being τ -vertical vector fields on $\tilde{\mathcal{E}}$. A covering $\tau: \tilde{\mathcal{E}} \rightarrow \mathcal{E}$ over a differentially connected equation is called irreducible if the covering equation $\tilde{\mathcal{E}}$ is differentially connected as well.

Symmetries, cosymmetries, conservation laws of the covering equation $\tilde{\mathcal{E}}$ are nonlocal symmetries, etc., of \mathcal{E} . Local objects depend on formal solutions of \mathcal{E} and their partial derivatives; roughly speaking, nonlocal ones may depend on integrals of these solutions⁴.

For example, the relations

$$\tilde{D}_x = D_x + \frac{wu}{2}, \quad \tilde{D}_t = D_t + \frac{w}{2} \left(\frac{u^2}{2} + u_x \right) \quad (8)$$

define a covering of the Burgers equation $\mathcal{E} = \{u_t = uu_x + u_{xx}\}$ by the heat equation $\tilde{\mathcal{E}} = \{w_t = w_{xx}\}$.

Thus, w is related to u by the formulas

$$w_x = \frac{wu}{2}, \quad w_t = \frac{w}{2} \left(\frac{u^2}{2} + u_x \right). \quad (9)$$

Note that system (9) is compatible by virtue of the Burgers equation.

The form $\omega = w dx + w_x dt$ is a local conservation law of $\tilde{\mathcal{E}}$, and its pullback to \mathcal{E} gives a nonlocal conservation law for \mathcal{E} . The corresponding nonlocal conserved density on \mathcal{E} , i.e., w , defined by (9), can be informally thought of as $\int \exp(\frac{u}{2}) dx$.

Going back to the general theory, note that any \mathcal{C} -differential operator Δ on \mathcal{E} can be lifted to a \mathcal{C} -differential operator $\tilde{\Delta}$ on $\tilde{\mathcal{E}}$ using equations (7). In particular, this can be done with the linearization operator $\ell_{\mathcal{E}}$ and its adjoint. Solutions of the equations

$$\tilde{\ell}_{\mathcal{E}}(\varphi) = 0, \quad \tilde{\ell}_{\mathcal{E}}^*(\psi) = 0$$

are called nonlocal shadows of symmetries and cosymmetries, respectively.

We employ the theory of coverings to establish nontriviality and independence of nonlocal conservation laws constructed in Section 4.

3 New coverings and nonisospectral Lax pair

We can readily write down a new covering for (2) expressed solely in terms of the variable $r = R(s)$:

Proposition 2. *The ABC equation (2) has a covering defined by the system*

$$\begin{aligned} r_t &= \frac{r((\kappa_1 + \kappa_2 - 1)(r + \kappa_1 + \kappa_2 - 1)u_{tx} - u_t r_x)}{u_x \kappa_1 (\kappa_1 + \kappa_2 - 1)}, \\ r_y &= \frac{r((\kappa_1 + \kappa_2 - 1)(r + \kappa_1 + \kappa_2 - 1)u_{xy} - \kappa_2 u_y r_x)}{u_x \kappa_1 (r + \kappa_1 + \kappa_2 - 1)}. \end{aligned} \quad (10)$$

Recall that a symmetry shadow (resp. a cosymmetry) for (2) is a solution of linearized (resp. adjoint linearized) version of (2), see Section 2 and [4, 22, 23, 24] for further details.

Proposition 3. *The ABC equation (2) has a shadow of nonlocal symmetry in the covering (10) with the characteristic*

$$U = \int (r + \kappa_1 + \kappa_2 - 1)^{\frac{(1-\kappa_1)}{(\kappa_1+\kappa_2-1)}} r^{\frac{(1-\kappa_2)}{(\kappa_1+\kappa_2-1)}} dr. \quad (11)$$

⁴Or, more generally, on solutions of some differential equations whose coefficients depend on formal solutions of \mathcal{E} .

Proposition 4. *The ABC equation (2) has a nonlocal cosymmetry with the characteristic*

$$\gamma = u_t \int (r + \kappa_1 + \kappa_2 - 1)^{\frac{-2\kappa_2}{(\kappa_1 + \kappa_2 - 1)}} r^{\frac{-2\kappa_1}{(\kappa_1 + \kappa_2 - 1)}} dr. \quad (12)$$

It can be shown that the shadow (11) cannot be lifted to a nonlocal symmetry for the ABC equation (2) in the covering (10).

Now pass from (10) to a slightly different (but equivalent) covering by putting $w = r u_x^{-(\kappa_1 + \kappa_2 - 1)/\kappa_1}$. Then we have

$$\begin{aligned} w_t &= -\frac{u_x^{(\kappa_2 - \kappa_1 - 1)/\kappa_1} w (\kappa_1 u_x u_t w_x + (\kappa_1 + \kappa_2 - 1) w (u_t u_{xx} - \kappa_1 u_x u_{xt}))}{\kappa_1^2 (\kappa_1 + \kappa_2 - 1)}, \\ w_y &= -\frac{\kappa_2 u_x^{(\kappa_2 - \kappa_1 - 1)/\kappa_1} w u_y (\kappa_1 u_x w_x + (\kappa_1 + \kappa_2 - 1) w u_{xx})}{\kappa_1^2 (u_x^{(\kappa_1 + \kappa_2 - 1)/\kappa_1} w + \kappa_1 + \kappa_2 - 1)}. \end{aligned} \quad (13)$$

Moreover, a similar change of variables in (5) leads to a significantly simpler Lax pair for (2):

Proposition 5. *The ABC equation (2) admits a nonisospectral Lax pair of the form*

$$\begin{aligned} \Psi_y &= \frac{\kappa_2 \lambda u_x^{(\kappa_2 - \kappa_1 - 1)/\kappa_1} u_y (-\kappa_1 u_x \Psi_x + (\kappa_1 + \kappa_2 - 1) \lambda u_{xx} \Psi_\lambda)}{\kappa_1^2 (u_x^{(\kappa_1 + \kappa_2 - 1)/\kappa_1} \lambda + \kappa_1 + \kappa_2 - 1)}, \\ \Psi_t &= \frac{\lambda u_x^{(\kappa_2 - \kappa_1 - 1)/\kappa_1} (-\kappa_1 u_x u_t \Psi_x + (\kappa_1 + \kappa_2 - 1) \lambda (u_t u_{xx} - \kappa_1 u_x u_{xt}) \Psi_\lambda)}{\kappa_1^2 (\kappa_1 + \kappa_2 - 1)}, \end{aligned} \quad (14)$$

which is related to (5) by the transformation $\Phi = \Psi$, $\lambda = u_x^{-(\kappa_1 + \kappa_2 - 1)/\kappa_1} R(\zeta/u_x)$.

Note that (14) can be obtained from (13) via the so-called Pavlov eversion, see [33, Section 2] and [13, 21] for details on the latter.

In stark contrast with (5), the variable spectral parameter λ enters (14) rationally. This enables us to construct an infinite sequence of conservation laws from (14) using a formal Taylor expansion for Ψ with respect to λ in the fashion outlined below.

4 Nonlocal conservation laws

Substituting into (14) a formal expansion $\Psi = \sum_{j=-\infty}^{\infty} \psi_j \lambda^j$ yields the equations

$$\begin{aligned} (\psi_j)_y &= \frac{\kappa_2 u_x^{(\kappa_2 - \kappa_1 - 1)/\kappa_1} u_y (-\kappa_1 u_x (\psi_{j-1})_x + (\kappa_1 + \kappa_2 - 1)(j-1) u_{xx} \psi_{j-1})}{\kappa_1^2 (\kappa_1 + \kappa_2 - 1)} - \frac{u_x^{(\kappa_1 + \kappa_2 - 1)/\kappa_1} (\psi_{j-1})_y}{(\kappa_1 + \kappa_2 - 1)}, \\ (\psi_j)_t &= \frac{u_x^{(\kappa_2 - \kappa_1 - 1)/\kappa_1} (-\kappa_1 u_x u_t (\psi_{j-1})_x + (\kappa_1 + \kappa_2 - 1)(j-1) (u_t u_{xx} - \kappa_1 u_x u_{xt}) \psi_{j-1})}{\kappa_1^2 (\kappa_1 + \kappa_2 - 1)} \end{aligned} \quad (15)$$

for $j \in \mathbb{Z}$.

However, the covering over (2) defined by (15) for $j \in \mathbb{Z}$ is pretty much intractable, and there appears to be no way to extract from it any reasonably simple conservation laws for (2).

Fortunately, the situation improves dramatically when we truncate the expansion for Ψ . One natural possibility to do this is to pass from the Laurent expansion for Ψ to the Taylor one, i.e., to assume that $\psi_j = 0$ for $j < 0$. The substitution of $\Psi = \sum_{j=0}^{\infty} \psi_j \lambda^j$ into (14) yields (15) for $j = 1, 2, 3, \dots$, and the equations

$$(\psi_0)_t = 0, \quad (\psi_0)_y = 0. \quad (16)$$

System (15) for $j = 1, 2, 3, \dots$ together with equations (16) defines an infinite-dimensional covering over (2) and yields, cf. e.g. [2, 3, 20, 35] and references therein, an infinite sequence of (in general nonlocal) two-component conservation laws for (2) of the form

$$\begin{aligned} \tilde{D}_y \left(\frac{u_x^{(\kappa_2 - \kappa_1 - 1)/\kappa_1} (-\kappa_1 u_x u_t (\psi_{j-1})_x + (\kappa_1 + \kappa_2 - 1)(j-1)(u_t u_{xx} - \kappa_1 u_x u_{xt}) \psi_{j-1})}{\kappa_1^2 (\kappa_1 + \kappa_2 - 1)} \right) = \\ \tilde{D}_t \left(\frac{\kappa_2 u_x^{(\kappa_2 - \kappa_1 - 1)/\kappa_1} u_y (-\kappa_1 u_x (\psi_{j-1})_x + (\kappa_1 + \kappa_2 - 1)(j-1) u_{xx} \psi_{j-1})}{\kappa_1^2 (\kappa_1 + \kappa_2 - 1)} - \frac{u_x^{(\kappa_1 + \kappa_2 - 1)/\kappa_1} (\psi_{j-1})_y}{(\kappa_1 + \kappa_2 - 1)} \right) \end{aligned} \quad (17)$$

for $j = 1, 2, 3, \dots$. The operators \tilde{D}_y and \tilde{D}_t denote here the total y - and t -derivatives in the covering (15), cf. e.g. [4, 24] and the discussion after Lemma 2 below for details. The nonlocal variables and the associated conservation laws are constructed recursively.

The simplest choice $\psi_0 = 0$, and even a more general choice $\psi_0 = \text{const}$, yields by virtue of (15)

$$(\psi_1)_t = 0, \quad (\psi_1)_y = 0.$$

If we now choose $\psi_1 = 1$, so $\Psi = \lambda + \sum_{j=2}^{\infty} \psi_j \lambda^j$, then we have

$$\begin{aligned} (\psi_2)_y &= \frac{\kappa_2 u_x^{(\kappa_2 - \kappa_1 - 1)/\kappa_1} u_y u_{xx}}{\kappa_1^2}, \\ (\psi_2)_t &= \frac{u_x^{(\kappa_2 - \kappa_1 - 1)/\kappa_1} (u_t u_{xx} - \kappa_1 u_x u_{xt})}{\kappa_1^2}, \end{aligned} \quad (18)$$

which gives rise to the first nontrivial conservation law in the sequence (17),

$$D_y \left(\frac{u_x^{(\kappa_2 - \kappa_1 - 1)/\kappa_1} (u_t u_{xx} - \kappa_1 u_x u_{xt})}{\kappa_1^2} \right) = D_t \left(\frac{\kappa_2 u_x^{(\kappa_2 - \kappa_1 - 1)/\kappa_1} u_y u_{xx}}{\kappa_1^2} \right). \quad (19)$$

This conservation law is local but the conservation laws (17) for $j = 3, 4, \dots$ will be nonlocal:

Proposition 6. *The ABC equation (2) has an infinite sequence of nontrivial nonlocal linearly independent conservation laws (17) for $j = 3, 4, \dots$, where $\psi_0 = 0$, $\psi_1 = 1$, the nonlocal variable ψ_2 is defined by (18), and the nonlocal variables ψ_j , $j = 3, 4, \dots$, are defined recursively via (15).*

Proof. We begin with a general construction. Let \mathcal{E} be a differentially connected equation and suppose that $\omega = A_1 dx^1 \wedge dx^3 \wedge \dots \wedge dx^n + A_2 dx^2 \wedge dx^3 \wedge \dots \wedge dx^n$ is a nontrivial two-component conservation law on $\tilde{\mathcal{E}}$, i.e.,

$$D_{x^2}(A_1) = D_{x^1}(A_2),$$

where D_{x^i} are the total derivatives on \mathcal{E} . Consider the covering $\tau_\omega: \mathcal{E}_\omega \rightarrow \mathcal{E}$ naturally associated to ω . This covering contains nonlocal variables ψ_σ , where σ is a multi-index whose components take values in the set $\{3, \dots, n\}$, that satisfy the defining equations

$$\begin{aligned} (\psi_\sigma)_{x^1} &= D_\sigma(A_1), \\ (\psi_\sigma)_{x^2} &= D_\sigma(A_2), \\ (\psi_\sigma)_{x^i} &= \psi_{\sigma i}, \end{aligned} \quad (20)$$

where $i = 3, \dots, n$.

Thus, infinitely many two-component conservation laws of the form

$$\omega_\sigma = D_\sigma(A_1) dx^1 \wedge dx^3 \wedge \dots \wedge dx^n + D_\sigma(A_2) dx^2 \wedge dx^3 \wedge \dots \wedge dx^n$$

arise on \mathcal{E} .

A straightforward generalization of the results proved in [19] is the following

Lemma 1. *If the equation \mathcal{E} is differentially connected then the conservation laws ω_σ are linearly independent if and only if the only solutions of the system*

$$\tilde{D}_{x^1}(f) = \tilde{D}_{x^2}(f) = 0$$

are functions of x^3, \dots, x^n , where \tilde{D}_{x^i} are the total derivatives on \mathcal{E}_ω .

From equations (20) it also immediately follows that under the assumptions of Lemma 1 the covering τ_ω is irreducible.

Now return to the ABC equation (1). Using the presentation (2) introduce the internal variables

$$u_k = \frac{\partial^k u}{\partial t^k}, \quad v_{k,l} = \frac{\partial^{k+l} u}{\partial t^k \partial x^l}, \quad w_{k,l} = \frac{\partial^{k+l} u}{\partial t^k \partial y^l}, \quad (21)$$

where $k \geq 0, l \geq 1$. The total derivatives in these coordinates take the form

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + \sum_k u_{k+1} \frac{\partial}{\partial u_k} + \sum_{k,l} \left(v_{k+1,l} \frac{\partial}{\partial v_{k,l}} + w_{k+1,l} \frac{\partial}{\partial w_{k,l}} \right), \\ D_x &= \frac{\partial}{\partial x} + \sum_k v_{k,1} \frac{\partial}{\partial u_k} + \sum_{k,l} \left(v_{k,l+1} \frac{\partial}{\partial v_{k,l}} + D_t^k D_y^{l-1}(\Upsilon) \frac{\partial}{\partial w_{k,l}} \right), \\ D_y &= \frac{\partial}{\partial y} + \sum_k w_{k,1} \frac{\partial}{\partial u_k} + \sum_{k,l} \left(D_t^k D_x^{l-1}(\Upsilon) \frac{\partial}{\partial v_{k,l}} + w_{k,l+1} \frac{\partial}{\partial w_{k,l}} \right), \end{aligned}$$

where Υ denotes the right-hand side of (2).

From the above formulas we immediately obtain

Lemma 2. *The ABC equation (2) is differentially connected.*

Now denote by Y_k the right-hand sides of first equations in systems (15) and (18) and by T_k the right-hand sides of second equations in these systems, $k \geq 2$. Then we have

$$\tilde{D}_y = D_y + \sum_{j \geq 2} \sum_{k \geq 0} \tilde{D}_x^k(Y_j) \frac{\partial}{\partial \psi_{j,k}}, \quad \tilde{D}_t = D_t + \sum_{j \geq 2} \sum_{k \geq 0} \tilde{D}_x^k(T_j) \frac{\partial}{\partial \psi_{j,k}},$$

where $\psi_{j,0} = \psi_j$ and $\psi_{j,k+1} = (\psi_{j,k})_x$.

The subsequent lemma is proved by direct computations using the expressions (18) and (15) for T_j :

Lemma 3. *The following estimates hold:*

$$\tilde{D}_x^k(T_2) = \frac{v_{0,1}^\alpha}{\kappa_1^2} (u_1 v_{0,k+2} - \kappa_1 v_{0,1} v_{1,k+1}) + o(2, k) \quad (22)$$

and

$$\tilde{D}_x^k(T_j) = (j-1) \frac{v_{0,1}^\alpha}{\kappa_1^2} (u_1 v_{0,k+2} - \kappa_1 v_{0,1} v_{1,k+1}) \psi_{j-1,0} - \frac{v_{0,1}^{\alpha+1} u_1}{\kappa_1 (\kappa_1 + \kappa_2 - 1)} \psi_{j-1,k+1} + o(j, k), \quad j > 2, \quad (23)$$

where $\alpha = (\kappa_2 - \kappa_1 - 1)/\kappa_1$, and $o(j, k)$ is a function that does not depend on the variables $\psi_{s,l}$ for $s > j-1$, $\psi_{j-1,l}$ for $l > k+1$, and $v_{0,l+2}$, $v_{1,l+1}$ for $l > k$.

We are now ready to establish a stronger result:

Lemma 4. *The only solutions of the equation*

$$\tilde{D}_t(f) = 0 \quad (24)$$

are functions of x and y .

Proof of Lemma 4. Suppose that

$$f = f(x, y, t, \dots, \psi_{2,0}, \dots, \psi_{2,k_2}, \dots, \psi_{j,0}, \dots, \psi_{j,k_j}) \quad (25)$$

is a solution of equation (24). Let us stress that in addition to x, y, t and $\psi_{j,k}$ the function f is allowed to depend on finitely many internal variables (21) on the ABC equation.

Now proceed by induction on j .

Base of induction: $j = 2$. Let $f = f(x, y, t, \dots, \psi_{2,0}, \dots, \psi_{2,k_2})$. Then from (22) one has

$$\tilde{D}_t(f) + \frac{v_{0,1}^\alpha}{\kappa_1^2} \sum_{k=0}^{k_2} ((u_1 v_{0,k+2} - \kappa_1 v_{0,1} v_{1,k+1}) + o(2, k)) \frac{\partial f}{\partial \psi_{2,k}} = 0.$$

We now perform induction on k_2 and show that f cannot depend on nonlocal variables $\psi_{2,0}, \dots, \psi_{2,k_2}$. Indeed, for $k_2 = 0$ one has

$$\tilde{D}_t(f) + \frac{v_{0,1}^\alpha}{\kappa_1^2} ((u_1 v_{0,2} - \kappa_1 v_{0,1} v_{1,1}) + o(2, 0)) \frac{\partial f}{\partial \psi_{2,0}} = 0.$$

But from the structure of the operator \tilde{D}_t it immediately follows that f is independent of $v_{0,2}$, and thus $\partial f / \partial \psi_{2,0} = 0$; hence the claim holds true. If $k_2 > 0$ then for similar reasons f cannot depend on v_{0,k_2+2} and consequently $f = f(x, y, t, \dots, \psi_{2,0}, \dots, \psi_{2,k_2-1})$. However, this form of f contradicts our initial assumption that $\partial f / \partial \psi_{2,k_2} \neq 0$, and hence f is local.

Induction step: $j > 2$. Let f be of the form (25). Then from the estimate (23) it follows that $k_{j-1} > k_j$; otherwise f will be independent of ψ_{j,k_j} . Repeating this argument, we obtain

$$k_2 > k_3 > \dots > k_{j-1} > k_j.$$

Using now the estimate (22), we see that the coefficient at $\partial / \partial \psi_{2,k_2}$ linearly depends on v_{0,k_2+2} , where $k_2 + 2 \geq k_j + j$. But this dependence is impossible for the reasons similar to those used in the proof of the case $j = 2$. Consequently, f is independent of all $\psi_{j,k}$, $k = 0, \dots, k_j$, and we arrive at the situation of the induction hypothesis, which completes the proof of Lemma 4. \square

From Lemma 4 it now follows that the only solutions of the equation $\tilde{D}_t(f) = \tilde{D}_y(f) = 0$ are functions of x alone, and the result of Proposition 6 now readily follows from Lemma 1. \square

5 Closing remarks

In the present paper we have constructed an infinite hierarchy of nonlocal conservation laws (17) for the ABC equation (2) and proved the nontriviality of those. An interesting byproduct of our work is the change of spectral parameter which simplifies the original nonisospectral Lax pair (5) down to (14).

Let us reiterate that the method of proving nontriviality of the nonlocal conservation laws in question is quite general and can be applied *mutatis mutandis* to many other multidimensional integrable systems.

In addition to being an important integrability attribute *per se*, the conservation laws found in our paper can be employed, by means of the associated potentials ψ_j , $j = 2, 3, \dots$, for the construction of nonlocal symmetries, nonlocal cosymmetries and further nonlocal conservation laws for the equation under study, cf., for example, [4, 7, 23, 24] and references therein.

On the other hand, just as the local conservation laws, the nonlocal ones give rise, once we perform a suitable change of independent variables and then rewrite our equation as an evolutionary system, to nonlocal integrals of motion, cf. e.g. [7, 32]. On the more speculative side, perhaps it could have been possible to apply the nonlocal conservation laws for the construction of exact solutions of the equation under study using a suitably adapted version of the method of conservation laws [18].

In closing note that in addition to the hierarchy of nonlocal conservation laws (17), all of which are two-component, the *ABC* equation (2) also admits *inter alia* a three-component conservation law of the form [5]

$$D_x((\kappa_1 - \kappa_2 + 1)u_y u_t) + D_y((1 - \kappa_1 + \kappa_2)u_x u_t) - D_t((\kappa_1 + \kappa_2 + 1)u_x u_y) = 0. \quad (26)$$

This shows, in particular, that the set of conservation laws (19), (17) for (2) is by no means complete.

In fact, acting on the conservation law (26) by appropriately chosen (nonlocal) symmetries, or even by shadows, is likely to yield many more (nonlocal) three-component conservation laws for (2). More broadly, finding non-two-component nonlocal conservation laws for multidimensional integrable systems is an interesting problem on its own right which is, however, beyond the scope of the present paper (cf. also the discussion in [7] and references therein for the systems which are not necessarily integrable).

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